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On the general solution to a certain class of heat and/or mass transfer problems

We would like to thank B. S. Baclic and P. J. Heggs for very valuable remarks presented in their letter to the Editors [1] on the paper [2]. The first part of their letter is concerned with the problems which we found necessary to respond to by presenting our common view. We hope that such a discussion will prove helpful in clearing up, and perhaps partly, in unifying the approach to a certain class of problems in application of heat and/or mass transfer theory. Most of the remarks presented in [1] had already been discussed in our private communication after [2] and [3] and before [4] were published. We are thus not surprised by the letter of Baclic and Heggs. It was our intention to make a similar review concerned with the methods applied to solutions of the problems postulated in the theory of heat and/or mass transfer, which are described by partial differential equations of the first degree as well as by Volterra's integral equations. So far only a small part of the project has been realized [5] and [6] in which the results achieved by Nusselt, Carslaw and Jaeger, Anzelius, Schumann (see refs. [2, 3, 18–20] of ref. [1]), Thomas [7], Brinkley [8] and Bonilla *et al.* [9] were discussed. More works within this scope are being advanced. These also include analysis of some references given in [1]. We acknowledge the fact that Baclic and Heggs quote the additional publications which should also be discussed.

The review of *various equivalent forms* of the solution of the Nusselt's model of the crossflow recuperator [1] is very valuable. We thank the authors for the effort they have made. A diversified approach to the foregoing problems (including the crossflow recuperator) stems from numerous trials in defining some special functions related to the Bessel functions. As far as we know, the authors of the special functions quoted in the letter did not deal with mathematical aspects of the boundary problem presented below. These authors, however, defined some special functions based only on the analysis of particular cases of the boundary problem. As to their Table 1, it should be supplemented at least by the works of Rabinovitch, whose results are presented in [10]. We should admit that we were not familiar with all the forms of the solution given by Baclic and Heggs, on the other hand one should mention here considerations by di Federico [11] which are also concerned with the problem discussed. However, the latter takes into account heat generation in one of the liquids. In [11] the set of Volterra's integral equations is solved.

With regard to the minor remark made by Baclic and Heggs in their letter, it is worth noting here that under some assumptions differential equations are frequently replaced by integral equations, which in such a case are just Volterra's equations. Next, one can then apply different versions of the fixed point theorem, of which those of Banach and Schauder are most commonly used. Such a treatment makes it possible to analyze both linear and nonlinear equations unlike these where operational calculus (including Laplace's transformation) are applied.

Before we go into more specific discussion we should admit that indeed in [2] there is a printing error in the argument of the exponential functions which was noticed in [1].

Further considerations will be refered to broader problems defined in the analyses of recuperators, regenerators, ion-exchange columns, chemical reactors, nuclear reactors etc. These problems are frequently undertaken anew, e.g. by Montakhab [12], and this has prompted us to elaborate [13].

It should be noted that the set of differential equations had already been solved [4]:

$$\begin{aligned}\frac{\partial u_1(x,y)}{\partial x} &= a_{11}u_1(x,y)+a_{12}u_2(x,y)+f_1(x,y),\\ \frac{\partial u_2(x,y)}{\partial y} &= a_{21}u_1(x,y)+a_{22}u_2(x,y)+f_2(x,y)\end{aligned} \tag{1}$$

with boundary conditions

$$u_1|_{x=0} = \varphi_1(y), \quad u_2|_{y=0} = \varphi_2(x) \tag{2}$$

where $u_j(x, y), j = 1, 2$, are unknown integrable (in Riemann's or Lebesgue's sense) functions, $f_j(x, y)$ are given integrable functions and $a_{jk}, j, k = 1, 2$, are prescribed constant coefficients.

The solution of the problem postulated in such a way was achieved by application of Mikusiński operational calculus. It should be noted that this method, which is justified on the basis of abstract algebra, does not impose any limitations on functions, except their integrability, whereas Laplace's transformation requires that the growth of the transformed function should not be rapid. Hence, the proven value as to the existence and uniqueness of Mikusiński's method is better, however, both methods prove that there exists a unique solution in the defined class of functions. Formal similarity of the Laplace's transformation and Mikusiński's operational calculus seems to be isomorphic. Notation introduced by Mikusiński is more convenient than that used in the Laplace's transformation. An interesting survey of the symbolic calculus is presented in [14]. In order to prove the existence and uniqueness of the solution in broader class of functions one should normally use methods based on the theories of differential and integral equations.

The solution of the problems (1) and (2) has been given in the form:

$$\begin{aligned}u_1(x,y) = {} & e^{a_{11}x}\varphi_1(y)+\int_0^y \varphi_1(\eta)\, e^{a_{11}x+a_{22}(y-\eta)}Bes_1(a_{12}a_{21}x, y-\eta)\, d\eta\\ & +a_{12}\int_0^x \varphi_2(\xi)\, e^{a_{11}(x-\xi)+a_{22}y}Bes_0[a_{12}a_{21}(x-\xi), y]\, d\xi\\ & +a_{12}\int_0^x\int_0^y f_2(\xi,\eta)\, e^{a_{11}(x-\xi)+a_{22}(y-\eta)}\, Bes_0[a_{12}a_{21}(x-\xi), y-\eta]\, d\eta\, d\xi\\ & +\int_0^x f_1(\xi,\eta)\, e^{a_{11}(x-\xi)}\, d\xi\\ & +\int_0^x\int_0^y f_1(\xi,\eta)\, e^{a_{11}(x-\xi)+a_{22}(y-\eta)}\, Bes_1[a_{12}a_{21}(x-\xi), y-\eta]\, d\eta\, d\xi,\end{aligned} \tag{3}$$

$$\begin{aligned}u_2(x,y) = {} & e^{a_{22}y}\varphi_2(x)+\int_0^x \varphi_2(\xi)\, e^{a_{11}(x-\xi)+a_{22}y}\, Bes_1(a_{12}a_{21}y, x-\xi)\, d\xi\\ & +a_{21}\int_0^y \varphi_1(\eta)\, e^{a_{11}x+a_{22}(y-\eta)}\, Bes_0[a_{12}a_{21}(y-\eta), x]\, d\eta\\ & +a_{21}\int_0^y\int_0^x f_1(\xi,\eta)\, e^{a_{11}(x-\xi)+a_{22}(y-\eta)}\, Bes_0[a_{12}a_{21}(y-\eta), x-\xi]\, d\xi\, d\eta\\ & +\int_0^y f_2(x,\eta)\, e^{a_{22}(y-\eta)}\, d\eta\\ & +\int_0^y\int_0^x f_2(\xi,\eta)\, e^{a_{11}(x-\xi)+a_{22}(y-\eta)}\, Bes_1[a_{12}a_{21}(y-\eta), x-\xi]\, d\xi\, d\eta,\end{aligned} \tag{4}$$

where the family of *simple special functions* of the two variables $Bes_n(x, y)$, $n = 0, +1, \ldots$, is defined by (3) as quoted in [1]. We should clearly emphasize, that it is always better if the special functions are of the simple form and appear in solutions in combination with elementary functions. It is important that the operation on special functions be easy. In the case under consideration this goal has been achieved. One can give a simple pattern (Fig. 1) illustrating the formation of each function both from Bes_n and Bs_n families. The latter ones are defined by a power series [2, equation (17)] and in natural way linked with Bes_n functions. The summing up of the elements situated diagonally or on the parallel lines (see Fig. 1) results in Bes_n function, where n denotes the line number given in Fig. 1. Function Bs_n, $n = 0, \pm 1, \ldots$, is defined as the sum of the elements situated on the nth line and on all lines denoted by numbers $> n$. Absolute convergence results from the fact, that the sum of all elements given in the table is equal to e^{x+y}. Hence, each series containing a part of these elements is also absolutely convergent.

The solution (3) and (4) has been verified by direct substitution to the set (1). The introduced special functions naturally result from operational forms of (3) and (4). In view of much interest on the part of industrial circles the solution (3) and (4) has also been verified in [15] by the method of successive approximations, which was applied to the set of Volterra's integral equations—equivalent to the problem (1) and (2). In the latter case it was shown that successive approximations form a sequence of partial sums of power series for the family of Bes_n functions. The method of successive approximations is thus one more proof of natural necessity for definition of these special functions. Theoretical aspects of differential equations in more general form than those considered here have been developed in [3, 15, 16]. There is no longer any doubt as to the existence and uniqueness of the solution. Considerations of a still more general nature are presented in [17]. They are aimed at elaboration of effective computer codes for resolving large sets of both linear and nonlinear Volterra's integral equations. These appear, among others, while models are worked-out for different heat and mass transfer procedures (i.e. multi-media or multi-pass heat exchangers, Field's recuperators, multipipe or Field's type fuel elements of nuclear reactors, etc.).

Special functions of $Bes_n(x, y)$ type are indeed of a very simple nature and fulfill a number of identities [13]. Exceptionally simple formulae for their differentiation as well as the first part of integration formulae have been also presented in [13].

$$
\begin{array}{c|ccccccc}
 & n=0 & 1 & 2 & m & k & \\
\hline
-1 & \frac{x^0 y^0}{0!\,0!} & \frac{x^1 y^0}{1!\,0!} & \frac{x^2 y^0}{2!\,0!} & \frac{x^m y^0}{m!\,0!} & \frac{x^k y^0}{k!\,0!} & \cdots \\
-2 & \frac{x^0 y^1}{0!\,1!} & \frac{x^1 y^1}{1!\,1!} & \frac{x^2 y^1}{2!\,1!} & & \frac{x^k y^1}{k!\,1!} & \cdots \\
-m & \frac{x^0 y^2}{0!\,2!} & \frac{x^1 y^2}{1!\,2!} & \frac{x^2 y^2}{2!\,2!} & & \frac{x^k y^2}{k!\,2!} & \cdots \\
 & \cdots & & & & & \cdots \\
-k & \frac{x^0 y^m}{0!\,m!} & \frac{x^1 y^m}{1!\,m!} & \cdots & \frac{x^m y^m}{m!\,m!} & \frac{x^k y^m}{k!\,m!} & \cdots \\
 & \cdots & & & & & \cdots \\
 & \frac{x^0 y^k}{0!\,k!} & \frac{x^1 y^k}{1!\,k!} & \cdots & \frac{x^m y^k}{m!\,k!} & \frac{x^k y^k}{k!\,k!} & \cdots \\
 & \cdots & \cdots & & & \cdots & \ddots
\end{array}
$$

FIG. 1.

Additional formulae based on [5] and [6] are given below:

$$\int e^{\alpha x} Bes_n(x,y)\,dx = (-\alpha)^{-(n+1)}\, e^{\alpha x} Bs_{n+1}(-\alpha x, -y/\alpha), \quad \alpha \neq 0 \tag{5}$$

$$\int e^{\alpha x} Bs_n(x,y)\,dx = \frac{e^{\alpha x}}{1+\alpha}[Bs_n(x,y)-(-\alpha)^n Bs_n(-\alpha x, -y/\alpha)], \quad \alpha \neq 0, \quad \alpha \neq -1 \tag{6}$$

$$I_n = \int e^{-x} Bs_n(x,y)\,dx = \begin{cases} I_0 - e^{-x}\sum_{k=1}^{n} Bs_k(x,y), & n = 1,2,\ldots \\ I_0 + e^{-x}\sum_{k=n+1}^{0} Bs_k(x,y), & n = -1,-2,\ldots \end{cases} \tag{7}$$

where

$$I_0 = e^{-x}[xBs_0(x,y) - yBs_2(x,y)]. \tag{8}$$

$$\int e^{\alpha x}\cos(\omega x-\varphi) Bes_n(\delta x, A)\,dx = \mathrm{Re}\left[\frac{1}{\delta}\left(-\frac{\beta}{\delta}\right)^{-(n+1)} e^{\beta x - i\varphi} Bs_{n+1}\left(-\beta x, -\frac{\delta}{\beta}A\right)\right], \quad \alpha\omega\delta \neq 0, \quad \beta = \alpha + i\omega, \quad i = \sqrt{-1} \tag{9}$$

$$\int e^{\alpha x}\cos(\omega x-\varphi) Bs_n(\delta x, A)\,dx = \mathrm{Re}\left\{\frac{1}{\delta+\beta} e^{\beta x - i\varphi}\left[Bs_n(\delta x, A) - (-\beta/\delta)^{-n} Bs_n\left(-\beta x, -\frac{\delta}{\beta}A\right)\right]\right\},$$
$$\alpha\omega\delta \neq 0, \quad \beta = \alpha + i\omega, \quad i = \sqrt{-1} \tag{10}$$

$$\int e^{\alpha y} Bes_n(x,y)\,dy = -(-\alpha)^{n-1}\, e^{\alpha y} Bs_n(-x/\alpha, -\alpha y), \quad \alpha \neq 0 \tag{11}$$

$$\int y\, e^{\alpha y} Bes_0(x,y)\,dy = \frac{1}{\alpha} e^{\alpha y}\left[yBes_0(x,y) - \frac{1}{\alpha}Bs_0(-x/\alpha, -\alpha y) + \frac{x}{\alpha^2}Bs_{-1}(-x/\alpha, -\alpha y)\right], \quad \alpha \neq 0 \tag{12}$$

$$\int y\, e^{\alpha y} Bes_1(x,y)\,dy = -\frac{x}{\alpha^2} e^{\alpha y} Bs_{-1}(-x/\alpha, -\alpha y), \quad \alpha \neq 0 \tag{13}$$

$$\int e^{\alpha y} Bs_n(x,y)\,dy = \frac{e^{\alpha y}}{1+\alpha}[Bs_n(x,y) - (-\alpha)^{n-1} Bs_n(-x/\alpha, -\alpha y)], \quad \alpha \neq 0, \quad \alpha \neq -1 \tag{14}$$

$$I_n = \int e^{-y} Bs_n(x,y)\,dy = \begin{cases} I_1 + e^{-y}\sum_{k=1}^{n-1} Bs_k(x,y), & n = 2,3,\ldots \\ I_1 - e^{-y}\sum_{k=n}^{0} Bs_k(x,y), & n = 0,-1,\ldots \end{cases} \tag{15}$$

where

$$I_1 = e^{-y}[yBs_1(x,y) - xBs_{-1}(x,y)]. \tag{16}$$

$$\int e^{\alpha y}\cos(\omega y-\varphi) Bes_n(A, \delta y)\,dy = \mathrm{Re}\left[-\frac{1}{\delta}\left(-\frac{\beta}{\delta}\right)^{n-1} e^{\beta y - i\varphi} Bs_n\left(-\frac{\delta}{\beta}A, -\beta y\right)\right], \quad \alpha\omega\delta \neq 0, \quad \beta = \alpha + i\omega, \quad i = \sqrt{-1} \tag{17}$$

$$\int e^{\alpha y} \cos(\omega y - \varphi) Bs_n(A, \delta y)\, dy = \mathrm{Re}\left\{\frac{1}{\delta+\beta} e^{\beta y - i\varphi}\left[Bs_n(A, \delta y) - (-\beta/\delta)^{n-1} Bs_n\left(-\frac{\delta}{\beta}A, -\beta y\right)\right]\right\},$$

$$\alpha\omega\delta \neq 0, \quad \beta = \alpha + i\omega, \quad i = \sqrt{-1}. \tag{18}$$

If in (9), (10), (17) and (18) the 'cos' function is replaced by 'sin' function, then the Re symbols should be replaced by Im, where Re and Im denote respectively the real and imaginary parts of a given complex number.

Introduction of Bes_n and Bs_n type special functions has made it possible to simplify several integral formulae required for the class of Bessel functions. Operations on Bes_n functions allow for their equivalent handling in all the quarters, whereas their presentation by means of the appropriate Bessel functions of I_n or J_n

$$Bes_n(x, y) \equiv \begin{cases} (x/y)^{n/2} I_n(2\sqrt{xy}), & xy > 0, \\ (-x/y)^{n/2} J_n(2\sqrt{-xy}), & xy < 0 \end{cases} \tag{19}$$

leads to the separate formulae for $xy > 0$ and $xy < 0$. Moreover, it becomes possible to present solutions of several particular cases of problems (1) and (2). This also includes the Nusselt's model of the cross-flow recuperator. The integral formulae mentioned and cited above make it possible to express those solutions in so-called 'closed form', i.e. without quadratures.

In view of the foregoing considerations it seems obvious that the presentation of the solution (19) in ref. [2] in the form as given in ref. [1, Table 1] cannot be recommended. This problem should rather be considered in a broader context as the one that constitutes a particular case of (1) and (2). Hence, this requires the application of Bs_n functions instead of the V_1 function, We regret that Baclic and Heggs while demonstrating the apparent equivalence of different forms of solutions, have considered only a partly generalized formulation of the problem. Perhaps they could be encouraged to the approach presented above by di Federico [11]. However, industrial circles are usually more interested in some particular solutions and hence, the problem is seldom postulated in a sufficiently general way. It is our understanding that if it had not been for Carslaw and Jaeger who followed the path set by Thomas [7], Brinkley [8] and others [1], and who scrutinized the problem (1), (2) (and not only its particular cases), not many new works would have appeared. We hope that the present discussion will prove helpful in making the solution (3) and (4) more popular.

It would be difficult not to agree with remarks concerning the cross-flow recuperator, since Baclic and Heggs have caught all the not too precise formulations in [2], and moreover, it should be stressed, they have thoroughly discussed this problem. They have also pointed out that a lot of additional information can be extracted from the analysis of the solution. However one has to remark, that the analysis of Baclic and Heggs may also be effectively done *using special functions* Bes_n *and* Bs_n.

Therefore we have limited ourselves to the presentation of some numerical results which will complement the analysis of Baclic and Heggs. The fluid temperature distributions calculated from equation (19) in [2] are shown in Figs. 2 and 3. The Tables 1 and 2 contain values of the mean outlet temperatures $\bar{\Theta}_1''$ and $\bar{\Theta}_2''$ respectively as functions of parameters b and R. The values of recuperator efficiency Φ_k as functions of parameters b, R and S, R are presented in the Tables 3 and 4, respectively. The notation is the same as that used in [2] and numerical results are given to four decimal places.

We hope that this discussion will contribute to the explanation of different approaches to the same mathematical problem. In this view the problem of the cross-flow recuperator with a constant heat transfer coefficient is a particular case of this general problem.

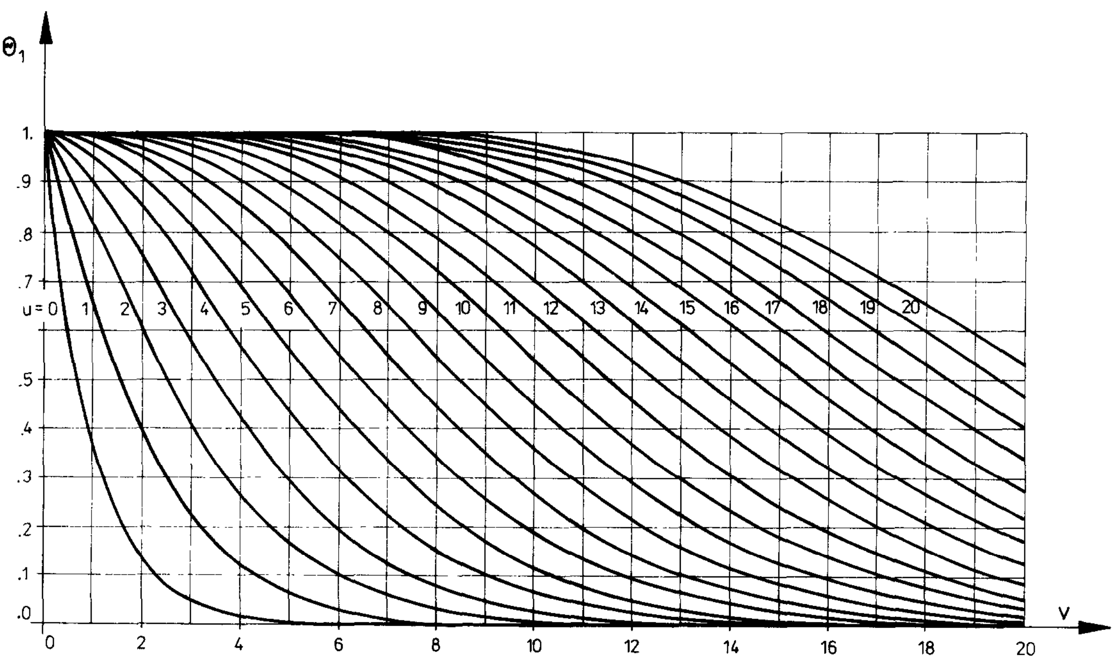

FIG. 2. Temperature distribution of the heating medium ($v = a \cdot \xi$, $u = b \cdot \eta$).

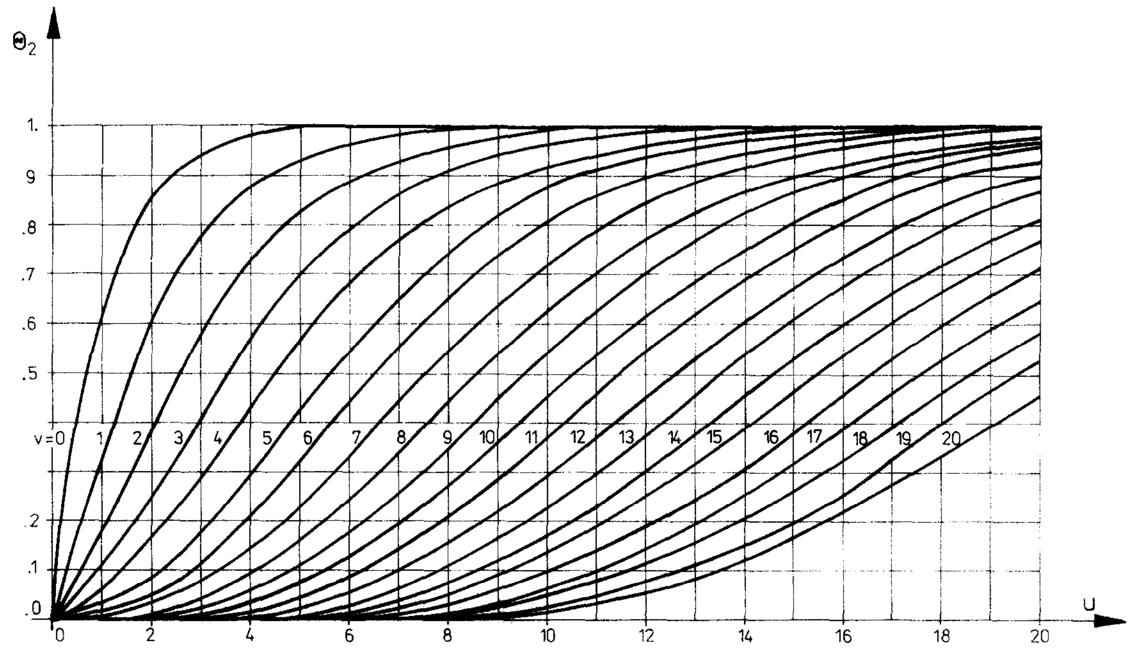

FIG. 3. Temperature distribution of the medium heated ($v = a \cdot \xi$, $u = b \cdot \eta$).

Table 1. The mean outlet temperature $\bar{\Theta}_1''$ as a function of the parameters b and R

	R										
b	0	0.2	0.4	0.6	0.8	1	1.5	2	3	4	5
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.2	1.0000	0.9644	0.9301	0.8969	0.8650	0.8342	0.7618	0.6958	0.5803	0.4839	0.4035
0.4	1.0000	0.9362	0.8764	0.8203	0.7678	0.7186	0.6089	0.5158	0.3697	0.2647	0.1892
0.6	1.0000	0.9136	0.8345	0.7620	0.6957	0.6350	0.5050	0.4011	0.2522	0.1579	0.0984
0.8	1.0000	0.8954	0.8013	0.7167	0.6406	0.5723	0.4308	0.3233	0.1806	0.0999	0.0548
1	1.0000	0.8807	0.7747	0.6807	0.5974	0.5238	0.3754	0.2676	0.1340	0.0660	0.0320
1.5	1.0000	0.8544	0.7273	0.6170	0.5217	0.4398	0.2837	0.1803	0.0702	0.0262	0.0095
2	1.0000	0.8378	0.6968	0.5756	0.4725	0.3858	0.2273	0.1303	0.0401	0.0115	0.0031
3	1.0000	0.8197	0.6609	0.5251	0.4116	0.3187	0.1605	0.0764	0.0152	0.0026	0.0004
4	1.0000	0.8110	0.6412	0.4953	0.3744	0.2776	0.1218	0.0489	0.0064	0.0007	0.0001
5	1.0000	0.8064	0.6292	0.4756	0.3490	0.2491	0.0965	0.0329	0.0029	0.0002	0.0000

Table 2. The mean outlet temperature $\bar{\Theta}_2''$ as a function of the parameters b and R

	R										
b	0	0.2	0.4	0.6	0.8	1	1.5	2	3	4	5
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.2	0.1813	0.1780	0.1749	0.1718	0.1688	0.1658	0.1588	0.1521	0.1399	0.1290	0.1193
0.4	0.3297	0.3192	0.3091	0.2995	0.2902	0.2814	0.2607	0.2421	0.2101	0.1838	0.1622
0.6	0.4512	0.4320	0.4138	0.3966	0.3804	0.3650	0.3300	0.2994	0.2493	0.2105	0.1803
0.8	0.5507	0.5228	0.4967	0.4722	0.4492	0.4277	0.3795	0.3383	0.2731	0.2250	0.1890
1	0.6321	0.5965	0.5633	0.5322	0.5033	0.4762	0.4164	0.3662	0.2887	0.2335	0.1936
1.5	0.7769	0.7280	0.6818	0.6384	0.5979	0.5602	0.4775	0.4099	0.3099	0.2434	0.1981
2	0.8647	0.8108	0.7580	0.7074	0.6593	0.6142	0.5152	0.4348	0.3200	0.2471	0.1994
3	0.9502	0.9016	0.8477	0.7915	0.7355	0.6813	0.5597	0.4618	0.3283	0.2493	0.1999
4	0.9817	0.9452	0.8969	0.8412	0.7819	0.7224	0.5854	0.4756	0.3312	0.2498	0.2000
5	0.9933	0.9680	0.9270	0.8740	0.8138	0.7509	0.6024	0.4835	0.3324	0.2500	0.2000

Table 3. The efficiency Φ_k of the cross-flow recuperator as a function of the parameters b and R

	R										
b	0	0.2	0.4	0.6	0.8	1	1.5	2	3	4	5
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.2	0.0000	0.0356	0.0700	0.1031	0.1350	0.1658	0.2382	0.3042	0.4198	0.5161	0.5965
0.4	0.0000	0.0638	0.1237	0.1797	0.2322	0.2814	0.3911	0.4842	0.6303	0.7354	0.8108
0.6	0.0000	0.0864	0.1655	0.2380	0.3043	0.3650	0.4950	0.5989	0.7478	0.8422	0.9016
0.8	0.0000	0.1046	0.1987	0.2833	0.3594	0.4277	0.5692	0.6767	0.8194	0.9001	0.9452
1	0.0000	0.1193	0.2253	0.3193	0.4026	0.4762	0.6246	0.7324	0.8660	0.9340	0.9680
1.5	0.0000	0.1456	0.2727	0.3830	0.4783	0.5602	0.7163	0.8197	0.9298	0.9738	0.9905
2	0.0000	0.1622	0.3032	0.4244	0.5275	0.6143	0.7728	0.8697	0.9599	0.9885	0.9969
3	0.0000	0.1803	0.3391	0.4749	0.5884	0.6813	0.8395	0.9236	0.9848	0.9974	0.9996
4	0.0000	0.1890	0.3588	0.5047	0.6256	0.7224	0.8782	0.9511	0.9936	0.9993	0.9999
5	0.0000	0.1936	0.3708	0.5244	0.6510	0.7509	0.9035	0.9671	0.9971	0.9998	1.0000

Table 4. The efficiency Φ_k of the cross-flow recuperator as a function of the parameters S and R

	R											
S	0	0.2	0.4	0.6	0.8	1	1.5	2	3	4	5	∞
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.2	0.0000	0.1193	0.1458	0.1565	0.1623	0.1658	0.1708	0.1733	0.1759	0.1772	0.1780	0.1813
0.4	0.0000	0.1622	0.2253	0.2543	0.2708	0.2814	0.2963	0.3042	0.3124	0.3166	0.3192	0.3297
0.6	0.0000	0.1803	0.2727	0.3193	0.3469	0.3650	0.3911	0.4051	0.4198	0.4273	0.4320	0.4512
0.8	0.0000	0.1890	0.3032	0.3650	0.4026	0.4277	0.4644	0.4842	0.5052	0.5161	0.5228	0.5507
1	0.0000	0.1936	0.3241	0.3987	0.4450	0.4762	0.5224	0.5475	0.5741	0.5880	0.5965	0.6321
1.5	0.0000	0.1981	0.3548	0.4534	0.5169	0.5602	0.6246	0.6597	0.6969	0.7162	0.7280	0.7769
2	0.0000	0.1994	0.3708	0.4864	0.5623	0.6143	0.6911	0.7324	0.7754	0.7974	0.8108	0.8647
3	0.0000	0.1999	0.3861	0.5244	0.6177	0.6813	0.7728	0.8197	0.8660	0.8885	0.9016	0.9502
4	0.0000	0.2000	0.3927	0.5456	0.6510	0.7224	0.8216	0.8697	0.9140	0.9340	0.9452	0.9817
5	0.0000	0.2000	0.3960	0.5590	0.6738	0.7509	0.8545	0.9017	0.9422	0.9591	0.9680	0.9933

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